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CIRCULARITY OF THE NUMERICAL RANGE OF ISOMETRICALLY BOUNDED LINEAR OPERATORS ON A HILBERT SPACE

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ABSTRACT

An equivalent condition on a non negative matrix with connected undirected graph so that its numerical range is a circular disc. Also using the adjacency matrix to determine the circularity of its numerical range.

Key words: Linear operator, numerical range, connected graph, circularity, adjacency matrix, Hilbert space.

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1 Introduction

Let A be an n-squarematrix. The numerical range of A is the set denoted by $W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$. By a result of Hausdorff and Toeplitz the numerical range is always a compact convex set and if $A \in \mathbb{C}_{2\times 2}$ then its numerical range is an elliptical disk with foci at the eigen values and minor axis of length $\sqrt{tr(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}$. Marcus and Pesce [2] gave a necessary and sufficient condition so that the numerical range W(A) of A is a circular disk centred at the origin for n=3 or 4. For n=3 the condition is $a_{12}a_{23}a_{13} = 0$, for n=4 it is $a_{12}a_{23}a_{13} + a_{12}a_{24}a_{14} + a_{13}a_{34}a_{14} + a_{23}a_{34}a_{24} = 0.$

When $A = (a_{r2})$ is an n-square real nilpotent upper triangular matrix then $a_{12}a_{13}a_{34}a_{14} = 0.$ In this paper we continue the study of circularity of numerical ranges in adjacency matrices of graphs and for a more generalized matrices. We know that every square complex matrix is unitarily similar to an upper triangular matrix and the numerical range of a matrix is unchanged if the matrix is transformed by a unitarily similarity. The introduction of graphs theory ideas in the study of numerical range seem to be uncommon. We will also find the sufficient conditions for the circularity of numerical ranges of adjacency matrices for a given graphs.

2 PRELIMINARIES

Defn 2.1 . A graph G is defined as a set of vertices and E is a set of edges. An edge is a pair of vertices more generally E is a pair (v_1, v_2) of two vertices.Edges can be either undirected or directed in which case (v_1, v_2) is distinct from (v_2, v_1). A graph that is directed can only be travelled one way on that particular edge, where as a graph without a direction can be travelled back and forth on the same edge without restrictions.

Defn.2.2. Let G be a graph with vertex set $\{v_1, v_2 .... v_n\}$. The adjacency matrix of G is an $n \times n$ matrix

$$A = (a_{ij}) \quad where\ a_{ij} = \begin{cases} 1\ if\ \ vertex\ v_i\ is\ adjacent\ to\ v_j \\ 0 \qquad\qquad\qquad otherwise \end{cases}$$

Proposition 2.3. For 2×2 matix A, a complete description of the numerical range W(A) is well known,namely W(A) is an ellipse with foci at the eigen values λ_1, λ_2 of A and a minor axis of the length $s = \left(trace(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{\frac{1}{2}}$, where S=0 for normal A and the ellipse in this case degenerates into a line segment connecting λ_1 *with* λ_2 and on the other hand for 2×2 matrix A with coinciding eigen values the ellipse W(A) degenerates into a disk.

Theorem 2.5. Let A be an $n\times n$ matrix with eigen values $\lambda_1 ....\lambda_n$ and suppose that its associated curve C(A) consists of k ellipses with minor axes of lengths $s_1, s_2, ..., s_k$ *and* $n-2k$ points then $\sum_{i-1}^{k} s_1^2 = trace\ \mathrm{A}^*\mathrm{A} - \sum_{i-1}^{n} |\lambda_1|^2$, for n=2 the condition imposed in the above theorem is of no restriction since C(A) is always an ellipse or a pair of points. If n=3 according to Kippenhahn's classification 2 the conditions of the theorem are satisfied.

Proof

Relabel the eigen values of A in such a way that $\lambda_{2i-1}, \lambda_{2i}$ become the foci of the j^{th} ellipse (i=1....k) and $\lambda_{2k+1} ...., \lambda_n$ the remaining points of C(A). Along with A, consider the matrix

$$\mathrm{B} = \begin{bmatrix} \lambda_1 & s_1 \\ 0 & \lambda_2 \end{bmatrix} \oplus \begin{bmatrix} \lambda_3 & s_2 \\ 0 & \lambda_4 \end{bmatrix} \oplus .... \oplus \begin{bmatrix} \lambda_{2k-1} & s_k \\ 0 & \lambda_{2k} \end{bmatrix} \oplus diag\ \lambda_{2k+1} ....\lambda_n$$

Since C(A)=C(B), the polynomials l_A *and* l_B have to be the same. If n=3 and A is in upper triangular form

$$\mathrm{A} = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & e \end{bmatrix}$$

We can have $|x|^2 c + |y|^3 b + |z|^3 a - x\overline{y}z = s^2 \lambda_3$, 2.5.1

where $s = \sqrt{|x|^2 + |y|^2 + |z|^2}$, 2.5.2

Hence conditions 2.5.1 , 2.5.2 are necessary for upper triangular matrices and

$$\mathrm{B} = \begin{bmatrix} \lambda_1 & s & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

To have the same associated curves and surprisingly a direct computation shows that they are sufficient.

Theorem 2.6

Let $\mathrm{A} = \begin{bmatrix} p & x & y \\ 0 & p & z \\ 0 & 0 & p \end{bmatrix}$ with p,x,y,z complex then

(1) W(A) is a disk if and only if xyz=0, in this case the disk has radius $\frac{1}{2}\sqrt{|x|^2+|y|^2+|z|^2}$ *with center* p.

(2) W(A) has a flat portion on its boundary if and only if $|x|=|y|=|z|>0$, in this case C(A) is a cardiod.

(3) W(A) is of the ovular shape if and only if $xyz \neq 0$ and $|x|,|y|,|z|$ are not all equal.

Part (1) for the nilpotent case was first shown by macus and pesce 3 , who developed a unitarily invariant form of the condition.

Part(2), consider the matrix 2A instead of A

$$2A = \begin{bmatrix} 2R & p & xy \\ \bar{x} & 2R\ \ p & z \\ \bar{y} & \bar{z} & 2R\ \ P \end{bmatrix} + i \begin{bmatrix} 2j\ \ p & -ix & -iy \\ -i\bar{x} & 2j\ \ p & -iz \\ i\bar{y} & i\bar{z} & 2i\ \ p \end{bmatrix}$$

By theorem 2.5 , W(A) has a flat portion if and only if there exist real u,v not both zero such that

$$|ux+v\ \ -ix\ \ | = |uy+v\ \ -iy\ \ | = |uz+v\ \ -iz\ \ |$$

$$\arg.\ ux-ivx\ \ +\arg.\ uz-ivz\ \ =\arg.\ uy-ivy$$

From the first equation we see that we must have $|u-iv||x|=|u-iv||y|=|u-iv||z|$ which implies that we must have $|u-iv||x|=|u-iv||y|=|u-iv||z|$ which implies that we must have $|x|=|y|=|z|$ since $u-iv \neq 0$.

The second equation becomes $\arg.\ u-iv\ \ =\arg.\ y\ \ -\arg.\ x\ \ -\arg.\ z$ where we can chose u,v so that this is true and the only condition we have is $|x|=|y|=|z|$.

We now prove that under this condition C(A) is a cardiod using unitary transformations $A \rightarrow U^*AU$ and multiplying A by scalars and suppose that its eigen value is 1/3, then $A = \begin{bmatrix} \frac{1}{3} & 1 & 1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

Using Fiedler,s formula 2 for the point equation of C(A) and transforming to polar co-ordinate we find

$3r^2\left(-3r-2+2\cos\theta\right)\left(-3r+2+2\cos\theta\right)=0$.

The factor of $3r^2$ is redundard since r=0 is a solution to the other two factors.The other two factors define the same curve, because if one replaces r with –r and θ *with* $\theta+\pi$, the factors are identical within a scalar multiple and in polar coordinate the factors trace the same curve. The equation therefore simplifies to $r=\frac{2}{3}\left(1-\cos\theta\right)$ which is an equation of a cardiod.

Part(3)

W(A) cannot be an ellipse without being a disk, the ovular shape is the only case left.

3 Adjacency matrix

Defn 3.1 Two graphs $G_1=\left(V_1,E_1\right)$ *and* $G_2=\left(V_2,E_2\right)$ are isomorphic if there is a one to one function $f:V_1\to V_2$ that preserves adjacency.By preserving adjacency, we mean for digraphs that for every pair of vertices v and w in V_1, $\left(v,w\right)$ *is in* E_1 *if only if* $\left(f\left(v\right),f\left(w\right)\right)$ *is in* E_2.i.e. $E_2=\left\{\left(f\left(v\right),f\left(w\right)\right):\left(v,w\right)\in E_1\right\}$

Defn 3.2 An invariantof graphs under isomorphism is a function g on graphs such that $g\left(G_1\right)=g\left(G_2\right)$ whenever G_1 *and* G_2 are isomorphic.

Defn.3.3 Let E be any binary relation on a finite set $v=\left\{v_1,......,v_n\right\}$.The adjacency matrix of E is the $n\times n$ Boolean matrix A defined by $\mathrm{A}\left(i,j\right)=1$ *if and only if* $\left(v_i,v_j\right)\in E$.

Example 3.3. The relation $\leq$ on the set $\left\{0,1,2,3,4\right\}$ is represented by the adjacency matrix

$$\mathrm{A}=\begin{bmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}$$

We note that all entries in the matrix above are 1 . This is because $\leq$ is reflexive and will be true for any reflexive relation.

Defn .3.4 Let S be any set and let $\oplus$ *and* $\otimes$ be any two binary operators defined on the elements of S.

Assume that $\oplus$ is associative. The inner product of $\oplus$ *and* $\otimes$, denoted by $\oplus\cdot\otimes$ is defined for $n\times n$ matrices over S by $A\oplus\cdot\otimes B = D$ such that $D(i,j) = (A(i,1)\otimes B(1,j))\oplus\ldots\oplus(A(i,n)\otimes B(n,j))$. By extention, when we wish to apply such an inner product to a single matrix, we write $(\oplus\cdot\otimes)^{k} A$ to denote the matrix A in the case that k=1 and for k>1 to denote the matrix $(\oplus\cdot\otimes)^{k-1} A\oplus.\otimes$ and for any single scalar operator $\oplus$ we will also write $A\oplus B$ for matrices A and B to denote the matrix E such that $E(i,j) = A(i,j)\oplus B(i,j)$.

(4) Conclussion

The motivation for this research was a question of circularity of numerical ranges from graphs and offer some sufficieent conditions

Theorem 4.1 (Marcus and Shure)

Let A be an nxn matrix (0,1) matrix having at most 1 in each row and column. Let $k_1 \ldots\ldots, k_n$ be the lengths of all circuits of G(A). Denote by v the longest length of a simple directed path in G(A) Which is not part of a circuit of G(A). Then the numerical range of A is given by $W(A) = conv\left[D(A)\cup P(A)\right]$ where D(A) is the circular disk centered at the origin with radius $\cos\left(\pi/_{v+2}\right)$ and P(A) is the polygon in the complex plane.

Furthermore, $W(A) = D(A)$ *if and only if* $G(A)$ contains no circuits.

Every n-square matrix is permutationally similar to a direct sum of square matrices with connected undirected graphs so without loss of generality the same is applied to connected undirected graphs in numerical ranges. If an adjacency complex matrix is such that its undirected graph is connected then the undirected graph is a tree and W(A) is a circular disk centered at the origin.

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